Proceedings of the American Academy of Arts and Sciences.

Vol. XLV. No. 13. — April, 1910.

CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY.

THE EFFECT OF LEAKAGE AT THE EDGES UPON THE TEMPERATURES WITHIN A HOMOGENEOUS LAMINA THROUGH WHICH HEAT IS BEING CONDUCTED.

By B. Osgood Peirce.

CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY.

THE EFFECT OF LEAKAGE AT THE EDGES UPON THE TEMPERATURES WITHIN A HOMOGENEOUS LAMINA THROUGH WHICH HEAT IS BEING CONDUCTED.

By B. Osgood Peirce.

Presented December 8, 1909. Received January 5, 1910.

In many of the determinations of thermal conductivity which have been made during the last few years, the so called "wall method" has been employed. That is, one face of a plate or wall of the material to be experimented upon has been kept at one constant temperature for a long time while the opposite face has been maintained at another constant temperature, and the quantity of heat per square centimeter of either face, which under these circumstances has passed per second from one face to the other, has been measured in some convenient way.

In practice such a plate is of limited dimensions, and although it is easy to insure that the temperatures of the faces shall be nearly uniform, it is comparatively difficult to maintain a steady gradient from face to face at the edges so that the heat flow within the slab shall be the same as if the faces were infinite in extent. If, however, the faces of the specimen to be used are small enough, it is possible to prevent almost entirely the escape of heat at the edges by surrounding the periphery by an arrangement like a Dewar flask. This is impracticable when for any reason the plate has to be large, and in this case it is necessary to make the thickness of the wall so small compared with the dimensions of the faces that the lines of flow of heat from face to face in the central portion of the slab shall not be appreciably distorted by loss of heat through the edges of the wall.

Some time ago, in an attempt to obtain an accurate average value of the conductivity of a given stratum in a certain deep mine, I had occasion to apply the wall method to some blocks of stone which were not perfectly homogeneous, and in order to represent the material fairly it seemed best to use a slab eight centimeters thick for each determination. The slabs were square and the edges were covered with lagging

to make the loss of heat through them as small as possible. Under these circumstances there was a very rough approximation to a uniform temperature gradient from the warm face to the cold one, at each edge, but it was difficult to measure the edge temperatures accurately and the areas of the faces were therefore made so large that the temperatures of points on the axis of the slab (that is, the line which joins the centres of the faces) would surely be the same within one one hundredth of a degree of the centigrade scale, in the final state, whether the whole of each edge was kept at the temperature of the warmer face or at the temperature of the colder face.

In anticipation of some further work of the same kind, I have been led to compute the final axial temperatures in a square slab ($a \times a \times c$) of thickness c, when one face is kept at temperature T_0 while the other face and all the edges are kept at the lower temperature T_1. The work is straightforward enough, but the computation when the slab is relatively broad is very laborious, and in view of the practical importance of the wall method in determinations of the conductivities of poor conductors of heat, it seems well to record some of the results.

The problem just stated is solved ($T_1 - WT_1 + WT_0$) when one has found [1] a solution (W) of the equation

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0 \tag{1}$$

which is equal to unity when $z = 0$, and to zero when $z = c$ for all positive values of x and y not greater than a; and which vanishes when $x = 0$, or $y = 0$, or $x = a$, or $y = a$, for all positive values of z not greater than c.

A convenient normal solution of (1) is

$$A\left(e^{\frac{k\pi z}{a}} - e^{\frac{k\pi(2c-z)}{a}}\right) \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{a}, \tag{2}$$

where $k^2 = m^2 + n^2$, and it is evident that

$$W(x, y, z) =$$

$$\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\left(\frac{16}{\pi^2 mn \sinh\frac{\pi kc}{a}} \cdot \sinh\frac{\pi k(c-z)}{a} \cdot \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{a}\right) \tag{3}$$

where m and n are odd integers.

[1] Byerly, Fourier's series, etc., p. 127.

The function

$$V = 1 - W(x, y, c - z), \tag{4}$$

which satisfies (1), is equal to unity when $z = 0$, and also for all positive values of z not greater than c, when $x = 0$, or $y = 0$, or $x = a$, or $y = a$. It vanishes when $z = c$, and the function

$$U = T_1 - W(T' - T_0) - V(T_1 - T') \tag{5}$$

or

$$T' - W(x, y, z)\cdot(T' - T_0) + W(x, y, c - z)\cdot(T_1 - T') \tag{6}$$

gives the temperatures in the slab if one face is kept at the temperature

TABLE I.

a	W
$\frac{1}{4}$ c	0.014
$\frac{1}{2}$ c	1.176
$\frac{2}{3}$ c	5.720
$\frac{4}{5}$ c	9.833
c	16.666
$\frac{3}{2}$ c	31.570
2 c	40.708
3 c	47.556
5 c	49.905

T_0, the other face at T_1, and the edges at T'. In an infinite slab of thickness c, the faces of which are kept at T_0 and T_1, the temperatures are given by the expression

$$U_\infty = (T_1 - T_0)\frac{z}{c} + T_0 \tag{7}$$

so that the difference between the values of the temperature at any point in the slab in the ideal case and the real case is

$$(T_1 - T_0)\left[\frac{z}{c} - W(x, y, c - z)\right] + (T' - T_0)\left[W(x, y, z) + W(x, y, c - z) - 1\right]. \tag{8}$$

The last factor of this expression has its maximum value at the middle point of the axis where $z = \frac{1}{2}c$.

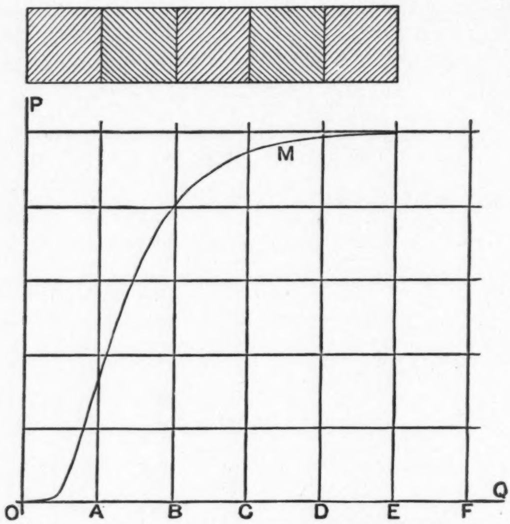

FIGURE 1. The ordinates of the curve show the temperatures, for different values of a, of a point Q in the centre of the axis (OS) of a square slab ($a \times a \times c$) of given thickness c, when one face ($a \times a$) is kept at the temperature 100° while the other face and the edges are kept at 0°. The horizontal unit is c, and it appears that when $a = 5\,c$, the temperature (49.9° +) of Q differs only slightly from the temperature (50°) which it would have if a were infinite. The shaded area above indicates the section of the slab for different values of a.

The value of W for the centre of the axis of the slab is given for several different values of a in Table I. When the ratio of a to c is large, the double series which defines W converges very slowly. Thus to obtain the last number in the table more than one hundred and fifty terms of the series were needed.

Figure 1 represents the numbers of Table I. graphically.

It is interesting to compare these results with similar ones for circular disks which Professor R. W. Willson and I obtained[2] several years ago.

---

[2] These Proceedings, 1898, **34**, 1.

TABLE II.

FINAL AXIAL TEMPERATURES IN A HOMOGENEOUS DISK OF DIAMETER d AND THICKNESS c, WHEN ONE FACE ($z = 0$) IS KEPT AT 100° C., THE OTHER FACE ($z = c$) AT 0° C., AND THE EDGE AT THE UNIFORM TEMPERATURE $\bar{\theta}$.

d/c	z/c	$\bar{\theta}=0°$	$\bar{\theta}=100°$	$\bar{\theta}=50°$
$\frac{1}{2}$	$\frac{1}{4}$	14.05	99.88	56.95
$\frac{1}{2}$	$\frac{1}{2}$	1.30	98.70	50.00
$\frac{1}{2}$	$\frac{3}{4}$	0.12	85.95	43.03
1	$\frac{1}{4}$	42.32	96.07	69.20
1	$\frac{1}{2}$	13.93	86.07	50.00
1	$\frac{3}{4}$	3.95	57.68	30.80
$\frac{3}{2}$	$\frac{1}{4}$	58.15	88.83	73.49
$\frac{3}{2}$	$\frac{1}{2}$	28.54	71.46	50.00
$\frac{3}{2}$	$\frac{3}{4}$	11.17	41.85	26.51
2	$\frac{1}{4}$	66.41	82.86	74.63
2	$\frac{1}{2}$	38.39	61.61	50.00
2	$\frac{3}{4}$	17.14	33.59	25.36
3	$\frac{1}{4}$	72.84	77.12	74.98
3	$\frac{1}{2}$	46.98	53.02	50.00
3	$\frac{3}{4}$	22.88	27.16	25.02
4	$\frac{1}{4}$	74.48	75.51	74.99
4	$\frac{1}{2}$	49.27	50.73	50.00
4	$\frac{3}{4}$	24.49	25.52	25.01
6	$\frac{1}{4}$	74.97	75.03	75.00
6	$\frac{1}{2}$	49.96	50.04	50.00
6	$\frac{3}{4}$	24.97	25.03	25.00
10	$\frac{1}{4}$	75.00	75.00	75.00
10	$\frac{1}{2}$	50.00	50.00	50.00
10	$\frac{3}{4}$	25.00	25.00	25.00

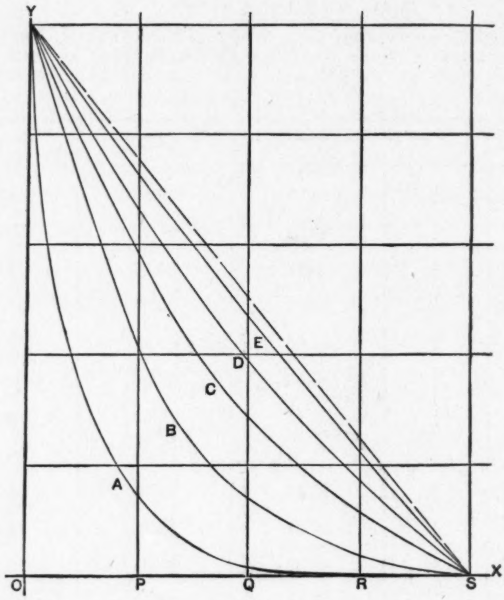

FIGURE 2. The curves show the final temperatures on the axis (OS) of a circular disk of given thickness (c) and of diameter d, when one face is kept at the temperature 100° and the other face and the rim at 0°. In A, B, C, D, and E, the diameter has the values $\frac{1}{2}c$, c, $\frac{3}{2}c$, $2c$, $3c$, respectively.

JEFFERSON PHYSICAL LABORATORY,
CAMBRIDGE, MASS.,
December, 1909.